Technical Notes

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Elastic Instability of Transversely Isotropic Timoshenko Beams

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Introduction

SEVERAL composite materials of current interest exhibit transverse isotropy† to such an extent that the ratio of the longitudinal Young's modulus to the transverse shear modulus (E/G^*) may be in the range from 20–50. This fact leads one to believe that shear deformation will play an important role in static, dynamic, and stability problems of transversely isotropic beams, plates, and shells.‡ In particular, it is the purpose of this Note to derive the buckling equations for a transversely isotropic Timoshenko beam and to present curves of buckling coefficients vs a parameter that measures the effect of shear deformation as a result of geometry and transverse isotropy. As will be seen, the effects are deleterious and can be large.

Buckling Equations

Assuming displacements of the form $u(x,z) = z\psi(x)$ and $w(x,z) = w(x)$, the pertinent strain-displacement relations are given by

$$\epsilon_x = zd\psi/dx \quad \epsilon_{xz} = \tfrac{1}{2}(\psi + dw/dx) \quad \epsilon_z = 0 \tag{1}$$

Defining the longitudinal modulus of elasticity to be E, the transverse shear modulus to be G^*, and Mindlin's shear correction factor to be κ^2 (which we will assume to be $\pi^2/12$),§ the stress-strain laws become upon inversion

$$\sigma_x = Ezd\psi/dx \quad \sigma_{xz} = \kappa^2 G^*(\psi + dw/dx) \tag{2}$$

Using Eq. (2), the shear and moment resultants are given by

$$Q_x = \int_{-h/2}^{h/2} b\sigma_{xz}dz = \kappa^2 G^* A^* \left(\psi + \frac{dw}{dx}\right) \tag{3}$$

$$M_x = -\int_{-h/2}^{h/2} b\sigma_x z dz = -EI\frac{d\psi}{dx} \tag{4}$$

where $A^* = bh$, $I = bh^3/12$, b is the beam width, and h is the beam depth. Referring to the deformed element in Fig. 1, the sum of the vertical forces and the y axis moments are given by

$$dQ_x/dx + N_x d^2w/dx^2 = 0 \tag{5}$$

$$dM_x/dx + Q_x = 0 \tag{6}$$

where N_x is the in-plane load. Putting Eqs. (3) and (4) into Eqs. (5) and (6) yields coupled equations for w and ψ. That is,

$$\kappa^2 G^* A^*(d\psi/dx + d^2w/dx^2) + N_x d^2w/dx^2 = 0 \tag{7}$$

$$-EI\frac{d^2\psi}{dx^2} + \kappa^2 G^* A^*\left(\psi + \frac{dw}{dx}\right) = 0 \tag{8}$$

After performing some elementary operations, the uncoupled counterparts to Eqs. (7) and (8) become

$$d^4w/dx^4 + k^2 d^2w/dx^2 = 0 \tag{9}$$

$$d^4\psi/dx^4 + k^2 d^2\psi/dx^2 = 0 \tag{10}$$

where

$$k^2 = (P/EI)/(1 - P/\kappa^2 G^* A^*) \text{ and } P = -N_x$$

The solutions of Eqs. (9) and (10), which satisfy the original Eq. (7) and (8), are then given by

$$w(x) = A\cos kx + B\sin kx + Cx + D \tag{11}$$

$$\psi(x) = k(1 - P/\kappa^2 G^* A^*)(A\sin kx - B\cos kx) - C \tag{12}$$

where A, B, C, and D are constants which must be determined by applying appropriate boundary conditions.

Types of Boundary Conditions

Simple support

The displacement and moment are zero and since $M_x \sim d\psi/dx$, one has $w = d\psi/dx = 0$ at the boundary.

Clamped support

The displacement and the rotation angle are zero so that

$$w = \psi = 0 \text{ at the boundary}$$

Free end

The moment ($\sim d\psi/dx$) is zero and since P is always horizontal (for the conservative load problem) it is seen that $P\,dw/dx = Q_x \equiv \kappa^2 G^* A^*(\psi + dw/dx)$ so that $d\psi/dx = (1 - P/\kappa^2 G^* A^*)\,dw/dx + \psi = 0$ at the boundary.

Solutions to Various Buckling Problems

I Cantilever beam

Applying the boundary conditions

$$w(0) = \psi(0) = \frac{d\psi(l)}{dx} = (P - \kappa^2 G^* A^*)\frac{dw(l)}{dx} - \kappa^2 G^* A^* \psi(l) = 0$$

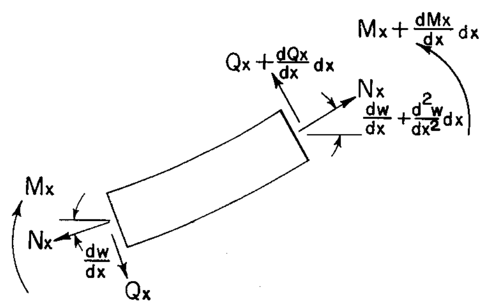

Fig. 1 Forces and moments on a deformed element.

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† For example, pyrolytic graphite, and unidirectional fibers (such as graphite, glass or boron) embedded in an epoxy matrix.

‡ Some isolated problems have been analyzed. For example, see Wu and Vinson,[1] Dudek,[2] and Brunelle.[3]

§ The correct determination of κ^2 has been discussed by Mindlin and Deresiewicz.[4]

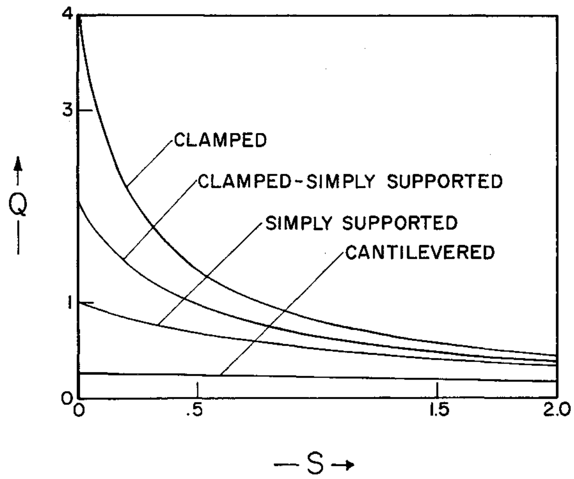

Fig. 2 Buckling coefficient Q vs S.

yields the following equations

$$A + D = 0 \quad -k(1 - P/\kappa^2 G^* A^*) B - C = 0$$

$$C = 0 \quad k^2(1 - P/\kappa^2 G^* A^*) (A \cos kl + B \sin kl) = 0$$

The classical buckling mode is found by setting $B = 0$ and hence $\cos kl = 0$ so that $k = \pi/2l$. Therefore solving $k^2 = (P/EI)/(1 - P/\kappa^2 G^* A^*)$ for P yields the following nondimensional buckling coefficient Q

$$Q = [4 + S]^{-1}$$

where

$$Q = (P/EI)\left(\frac{l}{\pi}\right)^2 \tag{13}$$

and

$$S = (E/G^*)\left(\frac{h}{l}\right)^2$$

If B is not taken as zero then a nonclassical buckling mode (pure shear mode) is given by

$$P_s = (\pi^2/12) G^* bh$$

but since the ratio P/P_s is always less than unity, Eq. (13) yields the lowest buckling coefficient.

II Simply Supported Beam

Applying the boundary conditions

$$w(0) = d\psi(0)/dx = w(l) = d\psi(l)/dx = 0$$

yields the following equations

$$B \sin kl + Cl = 0$$

$$B \sin kl = 0$$

which demands that $C = \sin kl = 0$ and, hence, $k = \pi/l$. Therefore Q is given by¶

$$Q = [1 + S]^{-1} \tag{14}$$

III Clamped Simply Supported Beam

Applying the boundary conditions

$$w(o) = \psi(o) = w(l) = d\psi(l)/dx = 0$$

yields the following equations

$$A \cos kl + B \sin kl = 0$$

$$[\cos kl - 1] A + [\sin kl - kl(1 - P/\kappa^2 G^* A^*)] B = 0$$

¶ In all cases $P/P_s < 1$ so that the pure shear mode will no longer be mentioned.

A nontrivial solution for A and B demands that

$$\tan kl = kl(1 - P/\kappa^2 G^* A^*)$$

Employing $a = kl/\pi$ the transcendental equation for a is given by

$$\tan a\pi = a\pi [1 + a^2 S]^{-1} \tag{15}$$

where a is related to Q by the result

$$Q = a^2 [1 + a^2 S]^{-1}$$

Table 1 provides Q vs S tabulations for Eq. (15).

IV Clamped Beam

Applying the boundary conditions

$$w(o) = \psi(o) = w(l) = \psi(l) = 0$$

yields the following equations

$$(\cos kl - 1) A + (\sin kl - kl[1 - P/\kappa^2 G^* A^*]) B = 0$$

$$(\sin kl) A + (1 - \cos kl) B = 0$$

Using the trigonometric identities

$$\sin kl = 2 \sin kl/2 \cos kl/2$$

$$\cos kl = 1 - 2 \sin^2 kl/2$$

a nontrivial solution for A and B demands that either

$$\sin kl/2 = 0$$

or that

$$\tan a\pi/2 = a\pi/2 [1 + a^2 S]^{-1}$$

The lowest value of k is found to be $k = 2\pi/l$ so that Q is given by

$$Q = 4[1 + 4S]^{-1} \tag{16}$$

Results

The results of Eqs. (13–16) are plotted in Fig. 2. Particularly note that as the boundary conditions become more restrained the effect of transverse shear (as a result of geometry and transverse isotropy) becomes more deleterious. Hence adding more boundary restraint does *not* substantially raise the buckling loads of even moderately transversely isotropic beams ($S > 0.5$) which is in sharp contrast to the behavior of a classical isotropic beam ($S = 0$) undergoing the same addition of boundary restraint.

Conclusions

It has been demonstrated that shear deformation due to geometry and transverse isotropy adversely affects the buckling loads of beams and that these effects become more pronounced as boundary restraint increases. This means that

Table 1 Q vs S for a clamped simply supported beam

S	Q
0	2.046
0.05	1.839
0.10	1.670
0.15	1.530
0.20	1.412
0.40	1.078
0.60	0.873
0.80	0.734
1.00	0.633
1.20	0.557
1.40	0.498
1.60	0.450
1.80	0.411
2.00	0.378

the analyst's intuition concerning the effectiveness of using boundary restraint to greatly increase buckling loads must be drastically modified when dealing with transversely isotropic materials.

References

[1] Wu, C. I. and Vinson, J. R., "On the Free Vibrations of Plates and Beams of Pyrolytic Graphite Type Materials," *AIAA Journal*, Vol. 8, No. 2, Feb. 1970, pp. 246–251.

[2] Dudek, T. J., "Young's and Shear Moduli of Unidirectional Composites by a Resonant Beam Method, *Journal of Composite Materials*, Vol. 4, April 1970, pp. 232–241.

[3] Brunelle, E. J., "The Statics and Dynamics of a Transversely Isotropic Timoshenko Beam," *Journal of Composite Materials*, Vol. 4, July 1970, pp. 404–416.

[4] Mindlin, R. D. and Deresiewicz, H., "Timoshenko's Shear Coefficient for Flexural Vibrations of Beams," *Proceedings of the Second U. S. National Congress of Applied Mechanics*, 1954, pp. 175–178.

Film Cooling Effectiveness with Helium and Refrigerant 12 Injection into a Supersonic Flow

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Nomenclature

c_p = specific heat
M = injection parameter, $\rho_2 U_2/\rho_\infty U_\infty$
Pr = Prandtl number
Re_2 = slot Reynolds number, $\rho_2 U_2 s/\mu_2$
s = length of porous section in flow direction
T = temperature
T_r = recovery temperature
T_* = reference temperature, [Eq. (2)]
U_∞ = velocity of mainstream in direction along the wall
U_2 = velocity of injectant at point of injection normal to the wall
W = molecular weight
x = distance along the wall downstream from the point of injection
ρ = density
μ = dynamic viscosity
η_{is} = high-speed flow effectiveness based on isoenergetic flow temperatures, [Eq. (1)]
ξ_* = dimensionless parameter, [Eq. (6)]

Subscripts

0 = condition in the stagnation chamber
2 = properties of injectant at point of injection
∞ = properties of mainstream
aw = adiabatic wall
$(\ )_{is}$ = designates isoenergetic flow condition
$*$ = evaluated at reference temperature T_*

Experimental measurements are reported for the film cooling effectiveness with normal injection (through a porous strip) of helium and refrigerant 12 into a two-dimensional turbulent boundary layer of a Mach 3 airflow. Related work has been reported in Refs. 1–4 (analytical studies) and Refs. 4–9 (experimental studies).

The apparatus has been described elsewhere.[4] The main flow of air has a Mach number of approximately 3 and a total temperature of about 295 K. The secondary flow (either helium or refrigerant 12) is subsonic and injected through a narrow porous section with an injection parameter $0.0018 < M < 0.0115$ and temperature $228\text{ K} < T_2 < 355\text{ K}$. A simple sketch of the geometry is included in Fig. 1.

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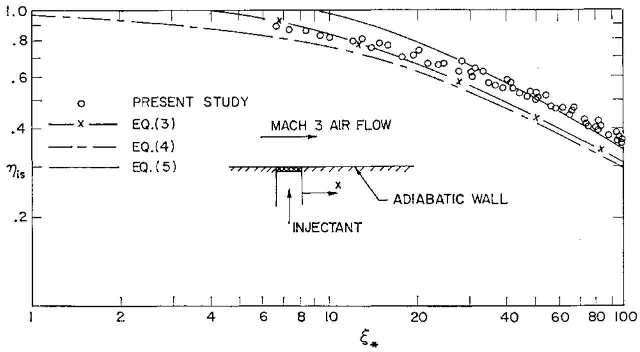

Fig. 1 Film cooling effectiveness with injection of helium for $M = 0.0018 - 0.0034$, $T_2 = 228 - 335°K$ {taking $c_{p\infty}/c_{p2} = 0.194$, and $Pr_{\infty *} = 0.728$ and $Re_2\ \mu_2/\mu_{\infty *} = 385$ [used in Eq. (5)]}.

Following earlier studies,[4,8] supersonic film cooling effectiveness is defined using isoenergetic flow temperatures

$$\eta_{is} = \frac{T_{aw} - (T_{aw})_{is}}{T_2 - (T_2)_{is}} \tag{1}$$

(Isoenergetic flow conditions are obtained with the same mainstream flow when the secondary gas, at the same rate of injection, has a total temperature equal to that of the main flow.) In order to eliminate the effect of day-to-day variation of the main-flow total temperature, the effectiveness is computed as[8]

$$\eta_{is} = \frac{(T_{aw}/T_0) - (T_{aw}/T_0)_{is}}{(T_2/T_0) - (T_2/T_0)_{is}} \tag{1a}$$

A reference state[4] is defined using

$$T_* = 0.28T_\infty + 0.72T_r \tag{2}$$

Modification of the earlier low-speed flow analyses to include effects of compressibility and foreign gas injection leads to the following equations for supersonic flow.[4,5,10] Stollery and El-Ehwany[2]

$$\eta_{is} = \{1 + (c_{p\infty}/c_{p2})[0.33\xi_*^{0.8} - 1]\}^{-1} \tag{3}$$

Kutateladze and Leont'ev[1]

$$\eta_{is} = \{1 + (c_{p\infty}/c_{p2})[0.33(4 + \xi_*)^{0.8} - 1]\}^{-1} \tag{4}$$

Goldstein and Haji-Sheikh[3]

$$\eta_{is} = \frac{1.9(Pr_{\infty *})^{2/3}}{1 + 0.33(c_{p\infty}/c_{p2})\xi_*^{0.8}(1 + \beta)} \tag{5}$$

with

$$\beta = 0.00015[\mathrm{Re}_2(\mu_2/\mu_{\infty *})](W_\infty/W_2) \tag{5a}$$

The parameter ξ_* in the foregoing equations is defined as

$$\xi^* = (x/Ms)[Re_2(\mu_2/\mu_{\infty *})]^{-0.25}(\rho_{\infty *}/\rho_\infty) \tag{6}$$

In Fig. 1 the experimental film cooling effectiveness for helium injection is plotted against the parameter ξ_*. On the same graph are shown the modified theoretical models. Data for refrigerant 12 injection are compared with the theoretical models in Fig. 2. Data using air injection were also obtained,[10] and the agreement of these with an earlier study[4] is satisfactory.

As indicated on the two graphs, the definition of effectiveness using isoenergetic flow temperatures and evaluation of the fluid properties at the reference temperature correlate the